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A BETTER APPROXIMATION OF DEFINITE INTEGRAL USING DIFFERENT NODES OF COMPOSITE TRAPEZOIDAL RULE

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Abstract

One of the major work of researchers in Mathematics especially in the area of Numerical, is to derive formulas that will give a better approximate solution to the exact solution of a particular problem. In this research work, the Composite Trapezoidal Rule (CTR) with 3, 5, 9 and 11 nodes were derived and applied to a particular integral problem. First the given problem was solved using the Simple Trapezoidal Rule (STR) and the result was used to compare the 3, 5, 9 and 11 nodes Composite Trapezoidal Rule (CTR). The results show that the Composite Trapezoidal rule yield a better approximation to the exact solution than the Simple Trapezoidal Rule (STR). The result further shows that the smaller the interval, the better the approximation of the integral to the exact solution.

Keywords: *Numerical Integration, Interpolation Quadrature, Trapezoidal rule, Composite Trapezoidal Rule.*

Introduction

Many numerical integral problems with unachievable solution when solved analytically can be solved numerically to get the approximate solution under a prescribed condition, (Mir, et al, 2020). In numerical, we seek the best method that yield a better approximation to the exact solution of any given problem. The quadrature represents an accurate method for numerical integration of polynomials and is a technique of subdividing the limit integral into equal intervals (Tahar, 2018).

Numerical integration also called numerical quadrature is a technique used in analysis to approximate the definite integral of a function over a specified interval. Numerical integration becomes necessary when standard mathematical methods cannot evaluate the integral, (Kress, 1998). Analytical integration, which works well with simple and well-defined function, falls short when dealing with complex real-world problems that lack closed-form solution. In such scenarios, numerical integration must be used as a way to approximate the integral's value to the correct level of accuracy (Kress, 1998). While trying to solve a higher order function, Abdulhameed and Qurban, (2021) in their study "An improved Trapezoidal Rule for Numerical Integration" developed a Composite Trapezoidal Rule which yielded a better approximate result in comparison to the existing simple trapezoidal rule. In addition to solving higher order functions, the Composite Trapezoidal Rule (CTR) could also be applied in different areas of engineering such as Aerospace, Civil, Marine, Mechanical, Automobile etc (Maharudra , Bheemsha & Rajanna, 2021). In the application of the method, Maharuda *et al*. (2021) used eight node elements with five degrees of freedom at each node. And the accuracy of their result was confirmed by comparing with available literatures. The approximate area of West Java Province was calculated with the use of Composite Trapezoidal Rule (Widal, et al 2021). In applying the rule, the province was partitioned into 9, 11 and 36 partitions with different quadrants and a better result was achieved.

There are two basic classes of the Newton-Cotes integration formulas, namely: the closed and open formulas. While the close formulas include the end-points of the range of integration among the nodes, the open formulas do not. The Trapezoidal and Simpson rules belong to the closed New-Cote integration formula.

The Simple Trapezoidal Rule

When $n=1$, the quadrature points are $x_0 \ and \ x_1$,there is one step with $x_0 = a \ and \ x_1 = b \ and \ h = b - a$. The quadrature rule is

$$\int_a^b f(x)dx = \sum_{k=0}^{1} C_k f_k = C_0 f(x_0) + C_1 f(x_1) \tag{1}$$

$= C_0 f(a) + C_1 f(b)$, from equation 1, we have the following

$$C_0 = h\int_0^1 \prod_{j\neq 0}^{1} \frac{s-j}{0-j} ds = h\int_0^1 \frac{s-1}{0-1} ds = -h\left(\frac{s^2}{2} - s\right)_0^1 = \frac{h}{2} \quad j \neq 0$$

$$C_0 = h\int_0^1 \prod_{j=0}^{1} \frac{s-j}{1-j} ds = h\int_0^1 \frac{s-1}{0-1} ds = -h\int_0^1 \frac{s-0}{1-0} = \frac{h}{2} \quad j \neq 1$$

The formula can therefore be written as

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a)+f(b)] \approx \frac{(b-a)}{2}[f(a)+f(b)] \tag{2}$$

Equation 2 above is known as the Simple Trapezoidal Rule (STR)

Derivation of Composite Trapezoidal Rule (CTR)

Peter (1994) observed that high accuracy is not often achieved with the simple quadrature rules (Simpson and Trapezoidal), hence the idea of the Composite Trapezoidal Rule (CTR) that give better approximation and high accuracy in comparison with the Simple Simpson and Trapezoidal rules.

A lower or simple quadrature rule can be made to possess higher accuracy by dividing its range of integration into numbers of n say of equal subintervals and applying the simple quadrature rule to each of the new subintervals. Thought the composite trapezoidal rule has been in existence with its benefit of yielding a better approximate result and high rate of accuracy, researchers find it difficult in using it because of its tedious computation. Since the smaller the intervals the better the approximation, there will be stage of computation before we arrive at the desire result. So, the research is to enlighten researchers of the important of the method and its wide application in real life situation.

$n = number\ of\ points$

$b = the\ upper\ \lim it\ of\ the\ range$

$a = lower\ \lim it\ of\ the\ range$

$h = is\ the\ step\ lenght\ which\ is\ defined\ as\ h = \frac{b-a}{a}$

Deriving the composite version of the Trapezoidal rule, we let $x_k = x_0 + kh,\quad x_0 = a,\quad x_n = b$, $x_k = x_{k-1} = h$ and $nh = b - a$.

The interval $[a, b]$ is divided to get $n+1$ nodes $x_0, x_2, x_3, \cdots, x_{n-1}, x_n$ in turn which yield n subintervals $[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3], \cdots, [x_{n-2}, x_{n-1}][x_{n-1}, x_n]$. The quadrature is then applied to each subinterval with

$$x_k = x_{k-1} = \frac{b-a}{n} = h,\quad k = 1, 2, 3, \cdots, n\ or\ x_k = x_0 + kh.$$

For the kth interval, $\left[x_{k-1}, x_k\right]$. The simple rule produces.

$$\int_{x_{k-1}}^{x_k} f(x)dx \approx \frac{h}{2}\left[f(x_{k-1}) + f(x_2)\right]$$

$$\int_{x_{k-1}}^{x_k} f(x)dx \approx \frac{h}{2}\left[f(x_0 + (k-1)h) + f(x_0 + kh)\right]$$

$$\int_{x_{k-1}}^{x_k} f(x)dx \approx \frac{h}{2}\left[f(a + (k-1)h) + f(a + kh)\right] \tag{3}$$

Now for $K = 1, \cdots, (n-1), n$ we get equation 4 from equation 3 after substitution,

$$\int_{x_{k-1}}^{x_k} f(x)dx \approx \frac{h}{2}\left\{\int_a^{x_1}\left[f(a) + f(a+h)\right] + \cdots + \int_{x_{n-2}}^{x_{n-1}}\left[f(a + (n-2)h) + f(a + (n-1))\right] + \int_{x_{n-1}}^{n}\left[(a + (n-1)h) + f(b)\right]\right\} \tag{4}$$

Now

$$\int f(a)dx \approx \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{n-2}}^{x_{n-1}} + \int_{x_{n-1}}^{x_1}\right) f(x)dx \tag{5}$$

$$\approx \frac{h}{2}\left\{\left[f(a) + f(a+h)\right] + \left[f(a+h) + f(a+2h)\right] + \cdots + \left[f(a + (n-2)h + f(a + (n-1)h))\right] + \left[f(a + (n-1)h) + f(b)\right]\right\} \tag{6}$$

Equation 6 can also be written as

$$\int_a^b f(x)dx \approx \frac{h}{2}\left\{f(a) + 2f(a+h) + f(a+2h) + \cdots + 2f(a + (n-1)h) + f(b)\right\} \tag{7}$$

And finally, the CTR in a compact form is written as follows

$$\int_a^b f(x)dx \approx \frac{h}{2}\left\{f(a) + f(b) + 2\sum_{k=1}^{n-1} f(a + kh)\right\} \tag{8}$$

Derivation of The Three Nodes Composite Trapezoidal Rule

Once the numbers of nodes is more than 2, the Simple Trapezoidal rule becomes composite. So, for nodes 3, we have the following variables for computation.

The nodes are $x_0, x_1, x_2,$ meaning $n+1=3,$ and $n-1=1.$ $h = \frac{x_2 - x_0}{2}$, since there are two subintervals, $[x_0, x_1]$ and $[x_1, x_2]$, the quadrature rule can therefore be written using equation 4 as follows

$k=1 \quad \int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\left[f(a)+f(a+h)\right]$

$k=2 \quad \int_{x_1}^{x_2} f(x)dx \approx \frac{h}{2}\left[f(a+h)+f(a+2h)\right]$, using the various nodes, we have

$\int_{x_0}^{x_2} f(x)dx \approx \left(\int_{x_0}^{x_1}+\int_{x_1}^{x_2}\right) f(x)dx$ substituting into equation 8, using the interval and the end-points, we have

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{4}\left[f(a)+f(a+h)+f(a+h)+f(b)\right] \qquad (9)$$

Equation 9 can be written in a compact form as follows:

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{4}\left[f(a)+f(b)+2f(a+h)\right] \qquad (10)$$

Derivation of the 5 Nodes Composite Trapezoidal Rule

Having 5 nodes, we therefore have the following

x_0, x_1, x_2, x_3, x_4, $n+1=5 \Rightarrow n=4$ and $h=\frac{x_4-x_0}{4}$, since we have four subintervals which include

$[x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4]$, we hereby have the following using equation 4

$k=1 \quad \int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\left[f(a)+f(a+h)\right]$

$k=2 \quad \int_{x_1}^{x_2} f(x)dx \approx \frac{h}{2}\left[f(a+h)+f(a+2h)\right]$,

$k=3 \quad \int_{x_2}^{x_3} f(x)dx \approx \frac{h}{2}\left[f(a+2h)+f(a+3h)\right]$

$k=4 \quad \int_{x_3}^{x_4} f(x)dx \approx \frac{h}{2}\left[f(a+3h)+f(a+4h)\right]$. In a similar way, we have

$\int_{x_0}^{x_4} f(x)dx \approx \left(\int_{x_0}^{x_1}+\int_{x_1}^{x_2}+\int_{x_2}^{x_3}+\int_{x_3}^{x_4}\right) f(x)dx$ and

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{h}{8}\left\{f(a)+f(b)+2f\left[(a+h)+(a+2h)+(a+3h)\right]\right\} \qquad (11)$$

Derivation of the 9 Nodes Composite Trapezoidal Rule

Having 9 nodes, we have the following variables for computation

$x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$, $n+1=9 \Rightarrow n=8$ and $h=\dfrac{x_8-x_0}{8}$, since we have eight subintervals which include

$[x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5], [x_5, x_6], [x_6, x_7], [x_7, x_8]$, we hereby have the following, using equation 4

$k=1 \quad \int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}[f(a)+f(a+h)] \quad k=2 \quad \int_{x_1}^{x_2} f(x)dx \approx \frac{h}{2}[f(a+h)+f(a+2h)]$,

$k=3 \quad \int_{x_2}^{x_3} f(x)dx \approx \frac{h}{2}[f(a+2h)+f(a+3h)] \quad k=4 \quad \int_{x_3}^{x_4} f(x)dx \approx \frac{h}{2}[f(a+3h)+f(a+4h)]$

$k=5 \quad \int_{x_4}^{x_5} f(x)dx \approx \frac{h}{2}[f(a+4h)+f(a+5h)] \quad k=6 \quad \int_{x_5}^{x_6} f(x)dx \approx \frac{h}{2}[f(a+5h)+f(a+6h)]$

$k=7 \quad \int_{x_6}^{x_7} f(x)dx \approx \frac{h}{2}[f(a+6h)+f(a+7h)] \quad k=8 \quad \int_{x_7}^{x_8} f(x)dx \approx \frac{h}{2}[f(a+7h)+f(a+8h)]$

In a similar way, we have

$\int_{x_0}^{x_8} f(x)dx \approx \left(\int_{x_0}^{x_1}+\int_{x_1}^{x_2}+\int_{x_2}^{x_3}+\int_{x_3}^{x_4}+\int_{x_4}^{x_5}+\int_{x_5}^{x_6}+\int_{x_6}^{x_7}+\int_{x_7}^{x_8}\right) f(x)dx$ and

$$\int_{x_0}^{x_8} f(x)dx \approx \frac{h}{16}\{[f(a)+f(a+h)]+[f(a+h)+f(a+2h)]+[f(a+2h)+f(a+3h)]+\cdots+[f(a+7h)+f(b)]\} \quad (12)$$

$$\int_{x_0}^{x_8} f(x)dx \approx \frac{h}{16}\{f(a)+f(b)+2f(a+h)+2f(a+2h)+2f(a+3h)+\cdots+2f(a+7h)\} \quad (13)$$

From equation 13 we can define the 9 nodes Composite Trapezoidal Rule as follows

$$\int_{x_0}^{x_8} f(x)dx \approx \frac{h}{16}\{f(a)+f(b)+2f[(a+h)+(a+2h)+(a+3h)+\cdots+(a+7h)]\} \quad (14)$$

Derivation of the 11 Nodes Composite Trapezoidal Rule

Having 11 nodes, we have the following points

$x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$, $n+1=11 \Rightarrow n=10$ and $h=\dfrac{x_{10}-x_0}{10}$, since we have ten subintervals which include

$[x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5], [x_5, x_6], [x_6, x_7], [x_7, x_8], [x_8, x_9], [x_9, x_{10}]$, we hereby have the following, using equation 8

$k=1 \quad \int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\left[f(a)+f(a+h)\right] \quad k=2 \quad \int_{x_1}^{x_2} f(x)dx \approx \frac{h}{2}\left[f(a+h)+f(a+2h)\right]$,

$k=3 \quad \int_{x_2}^{x_3} f(x)dx \approx \frac{h}{2}\left[f(a+2h)+f(a+3h)\right] \quad k=4 \quad \int_{x_3}^{x_4} f(x)dx \approx \frac{h}{2}\left[f(a+3h)+f(a+4h)\right]$

$k=5 \quad \int_{x_4}^{x_5} f(x)dx \approx \frac{h}{2}\left[f(a+4h)+f(a+5h)\right] \quad k=6 \quad \int_{x_5}^{x_6} f(x)dx \approx \frac{h}{2}\left[f(a+5h)+f(a+6h)\right]$

$k=7 \quad \int_{x_6}^{x_7} f(x)dx \approx \frac{h}{2}\left[f(a+6h)+f(a+7h)\right] \quad k=8 \quad \int_{x_7}^{x_8} f(x)dx \approx \frac{h}{2}\left[f(a+7h)+f(a+8h)\right]$

$k=9 \quad \int_{x8}^{x_9} f(x)dx \approx \frac{h}{2}\left[f(a+8h)+f(a+9h)\right] \quad k=10 \quad \int_{x_9}^{x_{10}} f(x)dx \approx \frac{h}{2}\left[f(a+9h)+f(a+10h)\right]$

In a similar way, we have

$\int_{x_0}^{x_{10}} f(x)dx \approx \left(\int_{x_0}^{x_1}+\int_{x_1}^{x_2}+\int_{x_2}^{x_3}+\int_{x_3}^{x_4}+\int_{x_4}^{x_5}+\int_{x_5}^{x_6}+\cdots+\int_{x_9}^{x_{10}}\right)f(x)dx$ and

$$\int_{x_0}^{x_{10}} f(x)dx \approx \frac{h}{20}\{[f(a)+f(a+h)]+[f(a+h)+f(a+2h)]+[f(a+2h)+f(a+3h)]+\cdots+[f(a+9h)+f(b)]\} \quad (15)$$

$$\int_{x_0}^{x_{10}} f(x)dx \approx \frac{h}{20}\{f(a)+f(b)+2f(a+h)+2f(a+2h)+2f(a+3h)+\cdots+2f(a+9h)\} \quad (16)$$

From equation 16 we can define the 11 nodes Composite Trapezoidal Rule as follows

$$\int_{x_0}^{x_{10}} f(x)dx \approx \frac{h}{20}\{f(a)+f(b)+2f[(a+h)+(a+2h)+(a+3h)+\cdots+(a+9h)]\} \quad (17)$$

Numerical Example

Given the integral $\int_0^1 \frac{dx}{1+x}$, we will apply it to the Simple Trapezoidal Rule with 2 nodes. And also, compare the result with that of the Composite Trapezoidal rule with node 3 5, 9 and 11.

The Simple Trapezoidal Rule

Using equation 2 above, the Simple Trapezoidal Rule, has the following approximate value

$a=0, \quad b=1, \quad f(x)=\frac{1}{1+x}, \quad f(a)=f(0)=1, \quad f(b)=f(1)=\frac{1}{2}, \quad h=\frac{1}{2}$

$$\int_0^1 \frac{1}{1+x}dx \approx \frac{1}{2}\left[1+\frac{1}{2}\right]=\frac{1}{2}\left[\frac{3}{2}\right]=\frac{3}{4} \approx 0.750000$$

The Three Nodes Composite Trapezoidal Rule

The Composite Trapezoidal Rule defined in equation 9, can be applied to the above problem as follows:

$$x_0, \quad x_1, \quad x_2. \quad h = \frac{b-a}{2} \Rightarrow h = \frac{1-0}{2} = \frac{1}{2}$$

$$x_0 = 0, \quad x_1 = a + h \Rightarrow x_1 = 0 + \frac{1}{2} = \frac{1}{2}, \quad x_2 = 1$$

$$f(0) = 1, \quad f\left(\frac{1}{2}\right) = \frac{2}{3}, \quad f(1) = \frac{1}{2}$$

Substituting the above values into equation 11 above, we have

$$\int_0^1 f(x)dx \approx \frac{1}{4}\left[1 + \frac{1}{2} + 2\left(\frac{2}{3}\right)\right] = \frac{17}{24} \approx 0.708333$$

Five Nodes Composite Trapezoidal Rule

Computing the 5 nodes, we have

$$x_0, \quad x_1, \quad x_2. \quad x_3, \quad x_4, \quad h = \frac{1}{4}$$

$$x_0 = 0, \quad x_1 = a + h \Rightarrow x_1 = \frac{1}{4}, \quad x_2 = \frac{2}{4} \quad x_3 = \frac{3}{4}, \quad x_4 = 1$$

$$f(0) = 1, \quad f\left(\frac{1}{4}\right) = \frac{4}{5}, \quad f\left(\frac{2}{4}\right) = \frac{4}{6}, \quad f\left(\frac{3}{4}\right) = \frac{4}{7}, \quad f(1) = \frac{1}{2}$$

Substituting the above values into the equation 11, we have

$$\int_0^1 f(x)dx \approx \frac{1}{8}\left[1 + \frac{1}{2} + +2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right)\right] \approx 0.697024$$

Nine Nodes Composite Trapezoidal Rule

With Nine nodes, we have the following data

$$x_0, x_1, x_2. x_3, x_4, x_5, x_6, x_7, x_8 \quad h = \frac{1}{8}$$

$$x_0 = 0, \quad x_1 = \frac{1}{8}, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{3}{8}, \quad x_4 = \frac{1}{2}, \quad x_5 = \frac{5}{8}, \quad x_6 = \frac{3}{4}, \quad x_7 = \frac{7}{8}, \quad x_8 = 1$$

Substituting into the given function, we have

$$f(0)=1,\ f\left(\frac{1}{8}\right)=\frac{8}{9},\ f\left(\frac{1}{4}\right)=\frac{4}{5}.\ f\left(\frac{3}{8}\right)=\frac{8}{11},\ f\left(\frac{1}{2}\right)=\frac{2}{3},\ f\left(\frac{5}{8}\right)=\frac{8}{13},\ f\left(\frac{3}{4}\right)=\frac{4}{7},\ f\left(\frac{7}{8}\right)=\frac{8}{15},\ f(1)=\frac{1}{2}$$

Substituting the above values into equation 13, we get the following result

$$\int_0^1 f(x)dx \approx \frac{1}{16}\left[1+\frac{1}{2}++2\left(\frac{6}{9}+\frac{4}{5}+\frac{8}{11}+\frac{2}{3}+\frac{8}{13}+\frac{4}{7}+\frac{8}{15}\right)\right]=\frac{1}{16}\left(\frac{45025075}{4054050}\right)\approx 0.694135$$

The Eleven Nodes Composite Trapezoidal Rule

With Eleven nodes, we have the following data

$$x_0,\ x_1,\ x_2.\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9,\ x_{10} \qquad h=\frac{1}{10}$$

$$x_0=0,\ x_1=\frac{1}{10},\ x_2=\frac{1}{5},\ x_3=\frac{3}{10},\ x_4=\frac{2}{5},\ x_5=\frac{1}{2},\quad x_6=\frac{3}{5},\quad x_7=\frac{7}{10},\ x_8=\frac{4}{5},\ x_9=\frac{9}{10},\ x_{10}=1$$

Substituting the above values into the given function, we have

$$x_0=0,\ x_1=\frac{10}{11},\ x_2=\frac{5}{6},\ x_3=\frac{10}{13},\ x_4=\frac{5}{7},\ x_5=\frac{2}{3},\quad x_6=\frac{5}{8},\quad x_7=\frac{10}{17},\ x_8=\frac{5}{9},\ x_9=\frac{10}{19},\ x_{10}=\frac{1}{2}$$

Substituting the above values into equation 17, we get the following result

$$\int_0^1 f(x)dx \approx \frac{1}{20}\left[1+\frac{1}{2}+\left(\frac{20}{11}+\frac{10}{6}+\frac{20}{13}+\frac{10}{7}+\frac{4}{3}+\frac{10}{8}+\frac{20}{17}+\frac{10}{9}+\frac{20}{19}\right)\right]=\frac{1}{20}\left(\frac{11{,}628{,}347{,}112}{838{,}053{,}216}\right)\approx 0.693771$$

The exact solution of the integral is 0.693147

Table 1. Presentation of Numerical Result

Nos of Nodes	*STR*	*CTR*
2	0.750000	
3		0.708333
5		0.697024
9		0.694135
11		0.693771

Conclusion

From the results in table 1 above, it can be shown that the 2 nodes Simple Trapezoidal Rule (STR) has an approximate value of 0.750000, which is not a good approximate value to the exact solution of the given integral. But considering column three of the above table, we observed that as the values of the nodes increase, the better the approximation of the function to the exact solution. This shows that the smaller the interval, the better the approximation. In real life application, area of irregular shape of land scape, convex bridge e.tc can easily be calculated with the help of Composite Trapezoidal Rule with good approximate result, which might not be achievable with simple Simpson Rule and the simple Trapezoidal Rule. Conclusively, most mathematical model of real life situation having many variables may not be solved with high accuracy of result using certain quadrature rule, in such situation, the composite trapezoidal rule should be the better option.

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